

The last part of this theorem may also be inferred from the preceding theorem. Constants  $\lambda_p$  and  $\Lambda_p$  for which (6) holds can be explicitly determined.

Schottky's theorem enables us to extend the above results for bounded functions to the case of functions  $f(z)$  omitting two values provided the sequence  $w_n = f(z_n)$  is bounded. The case  $|w_n| \rightarrow \infty$  can be treated by other methods.

<sup>1</sup> O. Szász, *Math. Zeitschrift*, **8**, 303-309 (1920).

<sup>2</sup> J. E. Littlewood, *Proc. London Math. Soc.*, **23**, 507 (1924); A. J. Macintyre, *Jour. London Math. Soc.*, **11**, 7-11 (1936).

## DIFFERENTIAL CALCULUS IN LINEAR TOPOLOGICAL SPACES<sup>1</sup>

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1. *Introduction.*—The most valuable definitions of differentials of functions in the classical differential calculi of finite as well as of infinite dimensional spaces are those that give the differential as a "first order approximation" to the difference. In this paper we give a definition of such a differential for functions whose arguments are in a linear topological space  $T_1$  and whose values are in a linear topological space  $T_2$ , not necessarily the same<sup>2</sup> as  $T_1$ . Some of the fundamental properties of this differential are given as well as the properties of other related topological differentials.

We wish to emphasize here the fact that the spaces  $T_1$  and  $T_2$  are not necessarily metric—not even metrizable—and that the differential calculus in linear topological spaces has important applications to general differential geometry, general dynamics and general continuous group theory.

2. *Topological M-Differential.*—By a linear topological space we shall mean an abstract linear space with a Hausdorff topology in which the functions  $x + y$  and  $\alpha x$  are respectively continuous functions of both variables.

Let  $T_1$  and  $T_2$  be any two linear topological spaces. A function  $l(x)$  on  $T_1$  to  $T_2$  is termed *linear* if it is additive and continuous—hence homogeneous of degree one.

DEFINITION OF M-DIFFERENTIAL.<sup>3</sup> Let  $f(x)$  be a function with values in  $T_2$  and defined on a Hausdorff neighborhood  $S_x$  of  $x_0 \in T_1$ . The function  $f(x)$  will be said to be *M-differentiable* at  $x = x_0$  and  $f(x_0; \delta x)$  will be called an *M-differential* of  $f(x)$  at  $x = x_0$  with increment  $\delta x$  if

(1) there exists a linear function  $f(x_0; \delta x)$  of  $\delta x$  with arguments in  $T_1$  and values in  $T_2$

(2) there exists a function  $\epsilon(x_0, x_1, x_2)$  with arguments in  $T_1$  and values in  $T_2$  such that

$$\begin{aligned} (a) \quad & \epsilon(x_0, 0, x) = 0 \text{ for all } x \in T_1 \\ (b) \quad & \epsilon(x_0, x_1, \lambda x_2) = \lambda \epsilon(x_0, x_1, x_2) \end{aligned}$$

for all  $\lambda > 0$ , for all  $x_1$  in some Hausdorff neighborhood of  $0 \in T_1$ , and for all  $x_2 \in T_1$

$$(c) \quad \epsilon(x_0, x_1, x_2)$$

is continuous in  $(x_1, x_2)$  at  $x_1 = 0, x_2 = x_2$  for all  $x_2 \in T_1$ .

(3) there exists some Hausdorff neighborhood  $S_0'$  of  $0 \in T_1$  such that for all  $\delta x \in S_0'$

$$f(x_0 + \delta x) - f(x_0) - f(x_0; \delta x) = \epsilon(x_0, \delta x, \delta x).$$

THEOREM 1. If an  $M$ -differential of  $f(x)$  at  $x = x_0$  exists, then it is unique and  $f(x)$  is continuous at  $x = x_0$ .

THEOREM 2. If  $f_1(x)$  and  $f_2(x)$  are  $M$ -differentiable at  $x = x_0$  then  $f_3(x) = \alpha f_1(x) + \beta f_2(x)$  is  $M$ -differentiable at  $x = x_0$  and

$$f_3(x_0; \delta x) = \alpha f_1(x_0; \delta x) + \beta f_2(x_0; \delta x).$$

THEOREM 3. Let  $T_3$  be a third linear topological space. If  $f(x)$  on  $S_{x_0} \subset T_1$  to  $T_2$  is  $M$ -differentiable at  $x = x_0$  and if  $\phi(y)$  on  $f(S_{x_0})$  to  $T_3$  is  $M$ -differentiable at  $y_0 = f(x_0)$ , then  $\psi(x) = \phi(f(x))$  is  $M$ -differentiable at  $x = x_0$  and

$$\psi(x_0; \delta x) = \phi(f(x_0); f(x_0; \delta x)).$$

3. OTHER DIFFERENTIALS AND THEIR RELATION TO THE  $M$ -DIFFERENTIAL. DEFINITION OF  $G$ -DIFFERENTIAL. Let  $f(x)$  be a function defined on a Hausdorff neighborhood  $S_{x_0}$  of  $x_0 \in T_1$  and with values in  $T_2$ . We shall say that  $f(x)$  is  $G$ -differentiable at  $x = x_0$  and  $f(x_0, \delta x)$  is its  $G$ -differential at  $x = x_0$  with increment  $\delta x$  if for any chosen  $\delta x \in T_1$ :

Given any Hausdorff neighborhood  $V_0$  of  $0 \in T_2$  there exists a  $\delta > 0$  such that<sup>4</sup>

$$\frac{f(x_0 + \lambda \delta x) - f(x_0)}{\lambda} \in f(x_0, \delta x) + V_0$$

for each  $\lambda$  satisfying  $0 < |\lambda| < \delta$ .

THEOREM 4. If  $f(x)$  is  $M$ -differentiable at  $x = x_0$ , then  $f(x)$  is  $G$ -differentiable at  $x = x_0$  and the two differentials are equal.

DEFINITION OF  $H$   $M$ -DIFFERENTIAL.<sup>5</sup> Let  $f(x)$  be a function with values in  $T_2$  and defined on a Hausdorff neighborhood  $S_{x_0}$  of  $x_0 \in T_1$ , and let  $x(\lambda)$  be any chosen function of a real variable  $\lambda$  with values in  $S_{x_0}$  and possessing a derivative  $\frac{dx(\lambda)}{d\lambda}$  at any chosen  $\lambda = \lambda_0$ . Write  $x_0 = x(\lambda_0)$ . The function  $f(x)$  will be said to be  $H$   $M$ -differentiable at  $x = x_0$  with  $f(x_0; \delta x)$  as its  $H$   $M$ -dif-

ferential at  $x = x_0$  if there exists a linear function  $f(x_0; \delta x)$  of  $\delta x$  having arguments in  $T_1$  and values in  $T_2$  such that for every admissible  $x(\lambda)$ :

$$(1) \quad \frac{d}{d\lambda} f(x(\lambda)) \text{ exists at } \lambda = \lambda_0$$

$$(2) \quad \frac{d}{d\lambda} f(x(\lambda)) = f\left(x_0; \frac{dx(\lambda)}{d\lambda}\right) \text{ for } \lambda = \lambda_0.$$

THEOREM 5. If  $f(x)$  is  $M$ -differentiable at  $x = x_0$ , then  $f(x)$  is  $H M$ -differentiable at  $x = x_0$  and the two differentials are equal.

THEOREM 6. If the linear topological spaces  $T_1$  and  $T_2$  are complete linear normed spaces (Banach spaces) and if  $f(x)$  is Fréchet differentiable<sup>6</sup> at  $x = x_0$ , then  $f(x)$  is  $M$ -differentiable at  $x = x_0$  and the two differentials are equal.

THEOREM 7. If the linear topological spaces  $T_1$  and  $T_2$  are finite dimensional arithmetic spaces and if  $f(x)$  is differentiable at  $x = x_0$  in the Stolz-Young-Fréchet sense, then  $f(x)$  is  $M$ -differentiable at  $x = x_0$  and the differentials are equal. Conversely if  $f(x)$  is  $M$ -differentiable at  $x = x_0$ , then it is differentiable in the Stolz-Young-Fréchet sense.

<sup>1</sup> Presented to the American Mathematical Society, April 9, 1938.

<sup>2</sup> In case  $T_1$  is a special linear topological space and  $T_2$  is the same space as  $T_1$ , a certain topological differential was defined by Michal and Paxson. It is still an open question whether the differentiability theorem on the composition of functions is valid for the Michal-Paxson differential. See Michal, A. D., and Paxson, E. W.: (1) "La Différentielle dans les Espaces Linéaires Abstraits avec une Topologie," *Comptes Rendus, Paris*, 202, 1741-1743 (1936); (2) "The Differential in Abstract Linear Spaces with a Topology," *Comptes Rendus de la Soc. de Sc. de Varsovie*, XXIX, 106-121 (1936).

<sup>3</sup> Another interesting type of differential can be defined by merely changing the equality relation in condition (3) of the definition for an  $M$ -differential into a set inclusion relation.

<sup>4</sup> By  $f(x_0, \delta x) + V_0$  we mean the set of all elements  $f(x_0, \delta x) + y$  as  $y$  ranges over the set  $V_0$ .

<sup>5</sup> A modified  $H M$ -differential is obtained if  $\lambda_0$  is always taken to be  $\lambda_0 = 0$ . This modified  $H M$ -differential is itself the abstraction of a differential for a function space studied recently by Fréchet. See page 244 of Fréchet, M., "Sur la Notion de Différentielle," *Journal de Math. Pures et Appl.*, 16, 233-250 (1937).

<sup>6</sup> Fréchet, M., "La Notion de Différentielle dans l'Analyse Générale," *Annales Ecole Norm.*, XLII, 293-323 (1925).